

# A Polynomial-time Algorithm for Outerplanar Diameter Improvement\*

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## Abstract

The OUTERPLANAR DIAMETER IMPROVEMENT problem asks, given a graph  $G$  and an integer  $D$ , whether it is possible to add edges to  $G$  in a way that the resulting graph is outerplanar and has diameter at most  $D$ . We provide a dynamic programming algorithm that solves this problem in polynomial time. OUTERPLANAR DIAMETER IMPROVEMENT demonstrates several structural analogues to the celebrated and challenging PLANAR DIAMETER IMPROVEMENT problem, where the resulting graph should, instead, be planar. The complexity status of this latter problem is open.

**Keywords:** diameter improvement; outerplanar graphs; completion problems; polynomial-time algorithms; dynamic programming; parameterized complexity.

## 1 Introduction

In general, a *graph completion problem* asks, given some input graph, whether it is possible to add edges to it so that the resulting graph satisfies some target property. There are two different ways of defining the optimization measure for such problems. The first, and most common, is the number of edges to be added, while the second is the value of some graph invariant on the resulting graph. Problems of the first type are HAMILTONIAN COMPLETION [16], INTERVAL GRAPH COMPLETION [18], PROPER INTERVAL GRAPH COMPLETION [17, 20], CHORDAL GRAPH COMPLETION [20, 24], and STRONGLY CHORDAL GRAPH COMPLETION [20], where the property is being Hamiltonian, interval, proper interval, chordal, and strongly chordal, respectively.

We focus our attention to the second category of problems where, for some given parameterized graph property  $\mathcal{P}_k$ , the problem asks, given a graph  $G$  and an integer  $k$ , whether it is possible to add edges to  $G$  such that the resulting graph belongs to

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$\mathcal{P}_k$ . Usually  $\mathcal{P}_k$  is a parameterized graph class whose graphs are typically required (for every  $k$ ) to satisfy some sparsity condition. There are few problems of this type in the bibliography. Such a completion problem is the PLANAR DISJOINT PATHS COMPLETION problem that asks, given a plane graph and a collection of  $k$  pairs of terminals, whether it is possible to add edges such that the resulting graph remains plane and contains  $k$  vertex-disjoint paths between each pair of terminals. While this problem is trivially NP-complete, it has been studied from the point of view of parameterized complexity [1]. In particular, when all edges should be added in the same face, it can be solved in  $f(k) \cdot n^2$  steps [1], i.e., it is fixed parameter tractable (FPT in short).

Perhaps the most challenging problem of the second category is the PLANAR DIAMETER IMPROVEMENT problem (PDI in short), which was first mentioned by Dejter and Fellows [9] (and made an explicit open problem in [12]). Here we are given a planar (or plane) graph and we ask for the minimum integer  $D$  such that it is possible to add edges so that the resulting graph is a planar graph with diameter at most  $D$  (according to the general formalism, for each  $D$ , the parameterized property  $\mathcal{P}_D$  contains all planar graphs with diameter at most  $D$ ). The computational complexity of PLANAR DIAMETER IMPROVEMENT is open, as it is not even known whether it is an NP-complete problem, even in the case where the embedding is part of the input. Interestingly, PLANAR DIAMETER IMPROVEMENT is known to be FPT: it is easy to verify that, for every  $D$ , its YES-instances are closed under taking minors<sup>1</sup> which, according to the meta-algorithmic consequence of the Graph Minors series of Robertson and Seymour [21, 22], implies that PLANAR DIAMETER IMPROVEMENT is FPT. Unfortunately, this implication only proves the *existence* of such an algorithm for each  $D$ , while it does not give any way to construct it. A uniform FPT algorithm for this problem remains as one of the most intriguing open problems in parameterized algorithm design. To our knowledge, when it comes to explicit algorithms, it is not even clear how to get an  $O(n^{f(D)})$ -algorithm for this problem (in parameterized complexity terminology, such an algorithm is called an XP-algorithm).

Notice that, in both aforementioned problems of the second type, the planarity of the graphs in  $\mathcal{P}_D$  is an important restriction, as it is essential for generating a non-trivial problem; otherwise, one could immediately turn a graph into a clique that trivially belongs to  $\mathcal{P}_1$ . For practical purposes, such problems are relevant where instead of generating few additional links, we mostly care about maintaining the network topology. The algorithmic and graph-theoretic study on diameter improvement problems has focused both on the case of minimizing the number (or weight) of added edges [2, 3, 6, 11, 14, 19], as well as on the case of minimizing the diameter [3, 15]. In contrast, the network topology, such as acyclicity or planarity, as a constraint to be preserved has received little attention in the context of complementing a graph; see for example [14].

In this paper we study the OUTERPLANAR DIAMETER IMPROVEMENT problem, or OPDI in short. An instance of OPDI consists of an outerplanar graph  $G = (V, E)$  and a positive integer  $D$ , and we are asked to add a set  $F$  of missing edges to  $G$  so that the resulting graph  $G' = (V, E \cup F)$  has diameter at most  $D$ , while  $G'$  remains outerplanar. Note that we are allowed to add arbitrarily many edges as long as the new graph is outerplanar. Given a graph  $G = (V, E)$ , we call  $G' = (V, E \cup F)$  a *completion* of  $G$ .

It appears that the combinatorics of OPDI demonstrate some interesting parallels with the notorious PDI problem. We denote by **opdi**( $G$ ) (resp. **pdi**( $G$ )) the minimum diameter of an outerplanar (resp. planar) completion of  $G$ . It can be easily seen that the treewidth of a graph with bounded **pdi**( $G$ ) is bounded, while the pathwidth of a graph with bounded **opdi**( $G$ ) is also bounded. In that sense, the OPDI can

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<sup>1</sup>To see this, if a graph  $G$  can be completed into an outerplanar graph  $G'$  of diameter  $D$ , then  $G'$  is also a valid completion of any subgraph  $H \subseteq G$ . Similarly, by merging two adjacent vertices  $uv$  in both  $G$  and  $G'$ , one sees that the diameter is also closed under edge contraction.

be seen as the “linear counterpart” of PDI. We stress that the same “small pathwidth” behavior of OPDI holds even if, instead of outerplanar graphs, we consider any class of graphs with bounded outerplanarity. Note also that both  $\mathbf{pdi}(G)$  and  $\mathbf{opdi}(G)$  are trivially 2-approximable in the particular case where the embedding is given. To see this, let  $G'$  be a triangulation of a plane (resp. outerplane) embedding of  $G$  where, in every face of  $G$ , all edges added to it have a common endpoint. It easily follows that, for both graph invariants, the diameter of  $G'$  does not exceed twice the optimal value.

Another closely related notion is that of  $t$ -spanners. Given a graph  $G = (V, E)$ , a  $t$ -spanner of  $G$  is a spanning subgraph  $G' = (V, E')$  such that for every pair of vertices, the distance in  $G'$  is within a factor  $t$  to their distance in  $G$ . The measure  $t$  for the quality of a spanner is called the *stretch factor*. The problem of identifying a  $t$ -spanner, or the  $t$ -SPANNER problem, has been extensively studied in geometry as well as in communication network design [4, 5, 7, 13, 23]. A sparse or *planar*  $t$ -spanner is of particular importance: in wireless ad hoc networks, certain routing protocols require the network topology to be planar [23]. The notion of  $t$ -spanner is, in a sense, dual to OPDI. The former allows losing the established links and the latter allows having additional links while, in both cases, our interest lies in preserving the constraint on the network topology. Note that in both cases, we want to improve (or do not deteriorate a lot) the performance of the network, which is measured by the lags in information transfer such as diameter or stretch factor.

**Our results.** In this work, we show that OUTERPLANAR DIAMETER IMPROVEMENT is polynomial-time solvable. Our algorithm, described in Section 2, is based on dynamic programming and works in full generality, even when the input graph may be disconnected. Also, our algorithm does *not* assume that the input comes with some specific embedding (in the case of an embedded input, the problem becomes considerably easier to solve).

**Further research.** Our algorithm for OPDI runs in time  $O(n^3)$  for connected input graphs, and in time  $O(n^7)$  or  $O(n^9)$  for disconnected input graphs, depending on whether  $D$  is odd or even, respectively, which can probably be improved.

We believe that our approach might be interesting for generalizations or variations of the OPDI problem, such as the one where we demand that the completed graph has fixed outerplanarity or is series-parallel.

By the Graph Minors series of Robertson and Seymour [21, 22] we know that for each fixed integer  $D$ , the set of *minor obstructions* of OPDI is *finite*. We have some preliminary results in this direction, but we managed to obtain a complete list only for small values of  $D$ . Namely, we obtained a partial list of forbidden substructures (not necessarily minimal), by using the notion of *parallel matching*. These partial results can be found in Appendix A.

Settling the computational complexity of PDI remains the main open problem in this area. An explicit FPT-algorithm, or even an XP-algorithm, would also be significant. Again, we have some partial NP-completeness result in this direction, but for a problem which is slightly more complicated than PDI, in particular involving edge weights. This reduction can be found in Appendix B.

## 2 Description of the algorithm

The aim of this section is to describe a polynomial-time dynamic program that, given an outerplanar graph  $G$  and an integer  $D$ , decides whether  $G$  admits an outerplanar completion with diameter at most  $D$ , denoted *diameter- $D$  outerplanar completion* for simplicity. Note that such an algorithm easily yields a polynomial-time algorithm to

compute, given an outerplanar graph  $G$ , the smallest integer  $D$  such that  $G$  admits a diameter- $D$  outerplanar completion.

Before describing the algorithm, we show some properties of outerplanar completions. In particular, Subsection 2.1 handles the case where the input outerplanar graph has cut vertices. Its objective is to prove that we can apply a *reduction rule* to such a graph which is safe for the OPDI problem. In Subsection 2.2 we deal with 2-vertex separators, and in Subsection 2.3 we present a polynomial-time algorithm for *connected* input graphs. Finally, we present the algorithm for disconnected input graphs in Subsection 2.4.

**Some notation.** We use standard graph-theoretic notation, see for instance [10]. It is well known that a graph is outerplanar if and only if it excludes  $K_4$  and  $K_{2,3}$  as a minor. An outerplanar graph is *triangulated* if all its inner faces (in an outerplanar embedding) are triangles. An outerplanar graph is *maximal* if it is 2-connected and triangulated. Note that, when solving the OPDI problem, we may always assume that the completed graph  $G'$  is maximal. A *block* in a graph is either a 2-connected component or an edge. Given a graph  $G$  and a subset  $S \subseteq V(G)$ , we denote by  $\partial(S)$  the set of vertices in  $S$  that have at least one neighbor in  $V(G) \setminus S$ .

## 2.1 Reducing the input graph when there are cut vertices

Given a graph  $G$ , let the *eccentricity* of a vertex  $u$  be  $\text{ecc}(u, G) = \max_{v \in V(G)} \text{dist}_G(u, v)$ . Given an outerplanar graph  $G$ , a vertex  $u \in V(G)$ , and an integer  $D$ , let us define  $\text{ecc}_D^*(u, G)$  as  $\min_H \text{ecc}(u, H)$ , where the minimum is taken over all the diameter- $D$  outerplanar completions  $H$  of  $G$ . If all the outerplanar completions have diameter more than  $D$ , we set this value to  $\infty$ . Unless said otherwise, we assume henceforth that  $D$  is a fixed given integer, so we may just write  $\text{ecc}^*(u, G)$  instead of  $\text{ecc}_D^*(u, G)$ . (The value of  $D$  will change only in the description of the algorithm at the end of Subsection 2.3, and in that case we will make the notation explicit).

As admitting an outerplanar completion with bounded eccentricity is a minor-closed property, let us observe the following:

**Lemma 1** *For any connected outerplanar graph  $G$ , any vertex  $v \in V(G)$ , and any connected subgraph  $H$  of  $G$  with  $v \in V(H)$ , we have that  $\text{ecc}^*(v, H) \leq \text{ecc}^*(v, G)$ .*

**Proof:** Let  $G'$  be an outerplanar completion of  $G$  achieving  $\text{ecc}^*(v, G)$ , that is such that  $\text{ecc}(v, G') = \text{ecc}^*(v, G)$ . Contracting the edges of  $G'$  that have at least one endpoint out of  $V(H)$  one obtains an outerplanar completion  $H'$  of  $H$  (as outerplanar graphs are minor-closed). As contracting an edge does not elongate any shortest path, we have that  $\text{dist}_{H'}(v, u) \leq \text{dist}_{G'}(v, u)$  for any vertex  $u \in V(H)$ , and in particular the diameter of  $H'$  is at most the diameter of  $G'$ , so  $\text{ecc}^*(v, H) < \infty$ . We thus have that  $\text{ecc}^*(v, H) \leq \text{ecc}(v, H') \leq \text{ecc}(v, G') = \text{ecc}^*(v, G)$ .  $\square$

Consider a connected graph  $G$  with a cut vertex  $v$ , and let  $C_1, \dots, C_t$  be the vertex sets of the connected components of  $G \setminus \{v\}$ . For  $1 \leq i \leq t$ , we call the vertex set  $B_i = C_i \cup \{v\}$  a *branch* of  $G$  at  $v$ . To shorten notations, we abbreviate  $B_i \cup \dots \cup B_j =: B_{i \dots j}$ , for any  $1 \leq i < j \leq t$ . Also, when referring to the eccentricity, we simply denote  $G[B_i]$  by  $B_i$ . Thus, for example, when considering the value  $\text{ecc}^*(v, B_{1 \dots i})$ , it will refer to the minimum eccentricity with respect to  $v$  that a diameter- $D$  outerplanar completion of the graph  $G[B_{1 \dots i}]$  can have.

The following lemma, which is crucial in order to obtain a polynomial-time algorithm, implies that if  $G$  has a cut vertex  $v$  with many branches, it is safe to remove most of them.

**Lemma 2** Consider an outerplanar graph  $G$  with a cut vertex  $v$  that belongs to at least 7 branches. Denote these branches  $B_1, \dots, B_t$ , with  $t \geq 7$ , in such a way that  $\text{ecc}^*(v, B_1) \geq \text{ecc}^*(v, B_2) \geq \dots \geq \text{ecc}^*(v, B_t)$ . The graph  $G$  has an outerplanar completion with diameter at most  $D$  if and only if  $\text{ecc}^*(v, B_{1\dots 6}) + \text{ecc}^*(v, B_7) \leq D$ .

**Proof:** “ $\Leftarrow$ ” If  $\text{ecc}^*(v, B_{1\dots 6}) + \text{ecc}^*(v, B_7) \leq D$ , gluing on  $v$  the outerplanar completions of  $G[B_{1\dots 6}], G[B_7], \dots, G[B_t]$ , respectively achieving  $\text{ecc}^*(v, B_{1\dots 6}), \text{ecc}^*(v, B_7), \dots, \text{ecc}^*(v, B_t)$ , one obtains a diameter- $D$  outerplanar completion  $G'$  of  $G$ . Indeed,

- The graph obtained is outerplanar and contains  $G$ .
- Two vertices  $x, y$  of  $G[B_{1\dots 6}]$  (resp. of  $G[B_i]$  for  $7 \leq i \leq t$ ) are at distance at most  $D$  from each other, as  $\text{ecc}^*(v, B_{1\dots 6}) < \infty$  (resp. as  $\text{ecc}^*(v, B_i) < \infty$ ).
- Any vertex  $x$  of  $G[B_{1\dots 6}]$  and  $y$  of  $G[B_i]$ , with  $7 \leq i \leq t$ , are respectively at distance at most  $\text{ecc}^*(v, B_{1\dots 6})$  and  $\text{ecc}^*(v, B_i) \leq \text{ecc}^*(v, B_7)$  from  $v$ . They are thus at distance at most  $\text{ecc}^*(v, B_{1\dots 6}) + \text{ecc}^*(v, B_7) \leq D$  from each other.
- Any vertex  $x$  of  $G[B_i]$  and  $y$  of  $G[B_j]$ , with  $7 \leq i < j \leq t$ , are respectively at distance at most  $\text{ecc}^*(v, B_i) \leq \text{ecc}^*(v, B_1) \leq \text{ecc}^*(v, B_{1\dots 6})$  (By Lemma 1) and  $\text{ecc}^*(v, B_j) \leq \text{ecc}^*(v, B_7)$  from  $v$ . They are thus at distance at most  $D$  from each other.

“ $\Rightarrow$ ” In the following, we consider towards a contradiction an outerplanar graph  $G$  admitting a diameter- $D$  outerplanar completion, but such that

$$\text{ecc}^*(v, B_{1\dots 6}) + \text{ecc}^*(v, B_7) > D. \quad (1)$$

Among the triangulated diameter- $D$  outerplanar completion of  $G$ , let  $G'$  be one that minimizes the number of edges added among distinct branches of  $G$ . Let us say that these edges are *lianas*. Let  $t'$  be the number of branches at  $v$  in  $G'$ , and denote these branches  $B'_1, \dots, B'_{t'}$ , in such a way that  $\text{ecc}^*(v, B'_1) \geq \text{ecc}^*(v, B'_2) \geq \dots \geq \text{ecc}^*(v, B'_{t'})$ . Then, since  $G'$  has diameter at most  $D$  and shortest paths among distinct branches of  $G'$  contain  $v$ ,

$$\forall_{1 \leq i' < j' \leq t'} \text{ecc}^*(v, B'_{i'}) + \text{ecc}^*(v, B'_{j'}) \leq D \quad (2)$$

is clear. Let  $S_{i'} := \{i \mid B_i \subseteq B'_{i'}\}$  for all  $1 \leq i' \leq t'$  and note that  $\{S_1, \dots, S_{t'}\}$  is a partition of  $\{1, \dots, t\}$ . The branches  $B'_{i'}$  with  $|S_{i'}| = 1$  are called *atomic*. Observe that an atomic branch is not incident with any liana. In contrast, for a non-atomic branch  $B'_{i'}$ , any  $B_i$  with  $i \in S_{i'}$  is incident to at least one liana in  $G'$  in order to connect the graph  $G'[B'_{i'} \setminus v]$ .

**Claim 1** Let  $B'_{i'}$  be a non-atomic branch and let  $S' \subsetneq S_{i'}$ . Then,  $\text{ecc}^*(v, \bigcup_{i \in S'} B_i) + \text{ecc}^*(v, \bigcup_{i \in S_{i'} \setminus S'} B_i) > D$ .

**Proof:** Let  $\mathcal{B} := \bigcup_{i \in S'} B_i$  and  $\bar{\mathcal{B}} := B'_{i'} \setminus \mathcal{B}$ . If the claim is false, then  $\text{ecc}^*(v, \mathcal{B}) + \text{ecc}^*(v, \bar{\mathcal{B}}) \leq D$ . Furthermore, for all  $j' \neq i'$ ,

$$\text{ecc}^*(v, \mathcal{B}) + \text{ecc}^*(v, B'_{j'}) \stackrel{\text{Lemma 1}}{\leq} \text{ecc}^*(v, B'_{i'}) + \text{ecc}^*(v, B'_{j'}) \stackrel{(2)}{\leq} D$$

and, likewise,  $\text{ecc}^*(v, \bar{\mathcal{B}}) + \text{ecc}^*(v, B'_{j'}) \leq D$ . Thus, the result of replacing  $G'[B'_{i'}]$  with the disjoint union of an outerplanar completion achieving  $\text{ecc}^*(v, \mathcal{B})$  and an outerplanar completion achieving  $\text{ecc}^*(v, \bar{\mathcal{B}})$  yields a diameter- $D$  outerplanar completion containing less lianas than  $G'$ , contradicting our choice of  $G'$ .  $\square$

**Claim 2** Let  $2 \leq i' \leq t'$ . Then, the branch  $B'_{i'}$  is atomic.

**Proof:** Towards a contradiction, assume that  $B'_{i'}$  is not atomic and let  $i \in S_{i'}$ . Then,

$$\begin{aligned} \text{ecc}^*(v, B'_{i'} \setminus (B_i \setminus v)) + \text{ecc}^*(v, B_i) &\stackrel{\text{Lemma 1}}{\leq} 2 \text{ecc}^*(v, B'_{i'}) \\ &\stackrel{i' > 1}{\leq} \text{ecc}^*(v, B_1) + \text{ecc}^*(v, B'_{i'}) \stackrel{(2)}{\leq} D, \end{aligned}$$

contradicting Claim 1.  $\square$

In the following, we abbreviate  $|S_1| =: s$ .

**Claim 3**  $S_1 = \{1, \dots, s\}$ .

**Proof:** First, assume  $1 \notin S_1$ . Then, by Claim 2, there is some  $i'$  with  $B'_{i'} = B_1$ . Let  $j'$  such that  $2 \in S_{j'}$ . Let  $G''$  denote the result of, for each  $j'$ , replacing  $G'[B'_{j'}]$  with an outerplanar completion of  $B_j$  achieving  $\text{ecc}^*(v, B'_{j'})$ . The diameter of  $G''$  is

$$\begin{aligned} \max_{i \neq j} \{ \text{ecc}^*(v, B_i) + \text{ecc}^*(v, B_j) \} &= \text{ecc}^*(v, B_1) + \text{ecc}^*(v, B_2) \\ &\stackrel{\text{Lemma 1}}{\leq} \text{ecc}^*(v, B'_{i'}) + \text{ecc}^*(v, B'_{j'}) \stackrel{(2)}{\leq} D, \end{aligned}$$

contradicting Equation (1).

Second, assume that there is some  $i \notin S_1$  with  $i+1 \in S_1$ . By Claim 2, there is some  $i' > 1$  with  $B'_{i'} = B_i$ . Then,

$$\begin{aligned} \text{ecc}^*(v, B'_1 \setminus (B_{i+1} \setminus v)) + \text{ecc}^*(v, B_{i+1}) &\stackrel{\text{Lemma 1}}{\leq} \text{ecc}^*(v, B'_1) + \text{ecc}^*(v, B_{i+1}) \\ &\leq \text{ecc}^*(v, B'_1) + \text{ecc}^*(v, B_i) \\ &\leq \text{ecc}^*(v, B'_1) + \text{ecc}^*(v, B'_{i'}) \stackrel{(2)}{\leq} D, \end{aligned}$$

contradicting Claim 1.  $\square$

**Claim 4** Let  $1 < i \leq s$ . Then,  $\text{ecc}^*(v, B_{1\dots i-1}) + \text{ecc}^*(v, B_i) > D$ .

**Proof:** Since by Claim 3,  $B_{1\dots i-1}$  and  $B_i$  are subsets of  $B'_1$ , this is a direct consequence of Claim 1 with  $S' = \{i, \dots, s\}$ .  $\square$

Note that, by (2),  $\text{ecc}^*(v, B_{1\dots s}) + \text{ecc}^*(v, B_{s+1}) \leq D$ . By (1), this implies  $s \geq 7$ . Thus, a special case of Claim 4 is  $\text{ecc}^*(v, B_1) + \text{ecc}^*(v, B_2) > D$ , implying  $\text{ecc}^*(v, B_1) > D/2$ .

**Claim 5** Let  $S' \subseteq \{1, \dots, t\}$  and let  $\mathcal{B} := \bigcup_{i \in S'} B_i$ . Then, there is a vertex in  $\mathcal{B}$  that is, in  $G'$ , at distance at least  $\text{ecc}^*(v, \mathcal{B})$  to any vertex of  $V(G) \setminus (\mathcal{B} \setminus v)$ .

**Proof:** Let  $H$  result from  $G'$  by contracting all vertices of  $V(G) \setminus \mathcal{B}$  onto  $v$ . If the claim is false, then for any vertex  $u \in \mathcal{B}$  there exists a vertex  $w \in V(G) \setminus (\mathcal{B} \setminus v)$  such that  $\text{dist}_{G'}(u, w) < \text{ecc}^*(v, \mathcal{B})$ , so contracting contracting all vertices of  $V(G) \setminus \mathcal{B}$  onto  $v$  yields a path in  $H$  between  $u$  and  $v$  of length strictly smaller than  $\text{ecc}^*(v, \mathcal{B})$ . As this argument holds for any vertex  $u \in \mathcal{B}$ , it implies that  $\text{ecc}(v, H) < \text{ecc}^*(v, \mathcal{B})$ . Since  $H$  is an outerplanar completion of  $G[\mathcal{B}]$ , this contradicts the definition of  $\text{ecc}^*$ .  $\square$

Two sub-branches  $B_i$  and  $B_j$  of  $B'_1$  are *linked* if  $G'$  has a liana linking them, that is, an edge intersecting both  $B_i \setminus \{v\}$  and  $B_j \setminus \{v\}$ .

**Claim 6** *Let  $1 \leq i < j \leq s$  and let  $\text{ecc}^*(v, B_{1\dots i}) + \text{ecc}^*(v, B_j) > D$ . Then,  $B_j$  is linked to one of  $B_1, \dots, B_i$ .*

**Proof:** By Claim 5, there is a vertex  $x \in B_j$  that is, in  $G'$ , at distance at least  $\text{ecc}^*(v, B_j)$  to any vertex in  $B_{1\dots i}$ . Likewise, there is a vertex  $y \in B_{1\dots i}$  that is, in  $G'$ , at distance at least  $\text{ecc}^*(v, B_{1\dots i})$  to any vertex in  $B_j$ . Let  $P$  be any shortest path of  $G'$  between  $x$  and  $y$ . By Claim 5, the maximal subpath of  $P$  in  $B_j \setminus v$  containing  $x$  has length at least  $\text{ecc}^*(v, B_j) - 1$  and the maximal subpath of  $P$  in  $B_{1\dots i} \setminus v$  containing  $y$  has length at least  $\text{ecc}^*(v, B_{1\dots i}) - 1$ . Since these subpaths are vertex disjoint and, by hypothesis,  $\text{ecc}^*(v, B_{1\dots i}) + \text{ecc}^*(v, B_j) > D$ , there is only one edge of  $P$  that is not contained in these subpaths and this edge intersects both  $B_{1\dots i}$  and  $B_j$ . This edge realizes the desired link.  $\square$

Claim 4 and Claim 6 clearly imply that for any  $1 \leq i < s$ ,  $B_{i+1}$  is linked to one of  $B_1, \dots, B_i$ . A consequence of the next claim will be that  $B_{i+1}$  is linked to exactly one of these branches.

**Claim 7** *The graph (with vertex set  $1, \dots, s$ ) induced by the relation linked ( $ij$  is an edge iff  $B_i$  is linked to  $B_j$ ) is a path. Furthermore, the subgraph induced by  $\{1, \dots, i\}$ , for any  $1 \leq i \leq s$ , is connected.*

**Proof:** The first statement is a consequence of the following three facts.

1. This graph is connected. Otherwise,  $v$  would be a cut vertex in  $G'[B'_1]$ , contradicting the definition of branch.
2. This graph has maximum degree 2. Consider for contradiction that some branch  $B_i$  is linked to three branches  $B_{j_1}$ ,  $B_{j_2}$ , and  $B_{j_3}$ . As each of  $B_i \setminus v$ ,  $B_{j_1} \setminus v$ ,  $B_{j_2} \setminus v$ , and  $B_{j_3} \setminus v$  induces a connected graph in  $G'$ , these four sets together with  $v$  induce a  $K_{2,3}$ -minor in  $G'$ , contradicting its outerplanarity.
3. This graph is not a cycle. Assume otherwise, so in particular it implies that  $s \geq 3$ . As each  $B_i \setminus v$  induces a connected graph in  $G'$ , these sets together with  $v$  would induce a  $K_4$ -minor in  $G'$ , contradicting its outerplanarity.

The second statement is a direct consequence of the fact that for any  $1 \leq i < s$ ,  $B_{i+1}$  is linked to one of  $B_1, \dots, B_i$ .  $\square$

Hence, for any  $1 \leq i \leq s$ , the graph  $G'[B_{1\dots i} \setminus v]$  is connected.

**Claim 8** *For any  $1 \leq i \leq s - 3$ ,  $\text{ecc}^*(v, B_{1\dots i+2}) > \text{ecc}^*(v, B_{1\dots i})$ .*

**Proof:** The monotonicity property given by Lemma 1 implies that  $\text{ecc}^*(v, B_{1\dots i+2}) \geq \text{ecc}^*(v, B_{1\dots i+1}) \geq \text{ecc}^*(v, B_{1\dots i})$ . Suppose for contradiction that  $\text{ecc}^*(v, B_{1\dots i+2}) = \text{ecc}^*(v, B_{1\dots i+1}) = \text{ecc}^*(v, B_{1\dots i}) =: c$ . Then, Claim 4 implies that  $c + \text{ecc}^*(v, B_j) > D$  for all  $j \in \{i+1, i+2, i+3\}$ . Thus, by Claim 6, each of  $B_{i+1}$ ,  $B_{i+2}$ , and  $B_{i+3}$  is linked to one of  $B_1, \dots, B_i$ . As each of  $B_{1\dots i} \setminus v$ ,  $B_{i+1} \setminus v$ ,  $B_{i+2} \setminus v$ , and  $B_{i+3} \setminus v$  induces a connected graph in  $G'$ , these sets together with vertex  $v$  induce a  $K_{2,3}$ -minor, contradicting the outerplanarity of  $G'$ .  $\square$

In the following let  $q$  be any integer such that  $4 \leq q \leq s$  and  $B_q$  is not linked to  $B_1$ . Note by Claim 8 that  $\text{ecc}^*(v, B_{1\dots q-1}) > \text{ecc}^*(v, B_1) > D/2$ . Let  $p < q$  be such that  $B_p$  and  $B_q$  are linked. By Claim 7,  $p$  is unique. In the following we study the structure of  $G'$  between these two branches.

As there is a liana between  $B_p$  and  $B_q$  and  $G'$  is triangulated, there is a path ( $v = x_0, x_1, x_2, \dots, x_r$ ), for some  $r \geq 1$ , of vertices of  $B_p$  (resp. a path ( $v = y_0, y_1, y_2, \dots, y_{r'}$ ), for some  $r' \geq 1$ , of vertices of  $B_q$ ) such that a vertex belongs to this path if and only if it is adjacent to a vertex of  $B_q \setminus \{v\}$  (resp.  $B_p \setminus \{v\}$ ) in  $G'$ . As  $G'$  is triangulated, it contains the edge  $\{x_1, y_1\}$ , and there are  $r + r' - 1$  lianas between  $B_p$  and  $B_q$ .

As for any  $0 < i < r$  there is a path from  $x_{i-1}$  to  $x_{i+1}$  whose inner vertices are  $y_j$ 's, note that in the graph  $G'[B_{1\dots q-1}]$ , the blocks containing the edges  $x_{i-1}x_i$  and  $x_i x_{i+1}$  are distinct. Indeed, otherwise the vertices  $x_{i-1}, x_i$ , and  $x_{i+1}$  together with a path between  $x_{i-1}$  and  $x_{i+1}$  in  $G'[B_{1\dots q-1}]$ , plus the vertex resulting from the contraction of  $B_q$  would induce a  $K_4$ -minor, contradicting the outerplanarity of  $G'$ . Let  $X$  be the block of  $G'[B_{1\dots q-1}]$  containing the edge  $vx_1$ . Note that every branch  $B_i$  with  $i < q$  and  $i \neq p$  belongs to  $X$ . Similarly the blocks containing the edges  $y_i y_{i+1}$  are all distinct.

**Claim 9** *In  $G'$ , there is no vertex of  $B_{1\dots q-1}$  at distance more than  $\text{ecc}^*(v, B_{1\dots q-1})$  to any vertex of  $B_q$ .*

**Proof:** Suppose there exists such a vertex  $x$ . As by Claim 5 some vertex  $y$  of  $B_q$  is at distance  $\text{ecc}^*(v, B_q)$  to any vertex of  $B_{1\dots q-1}$ , the two vertices  $x, y$  would be at least  $\text{ecc}^*(v, B_{1\dots q-1}) + \text{ecc}^*(v, B_q)$  apart. By Claim 4 this distance is more than  $D$ , a contradiction.  $\square$

**Claim 10** *Let  $x \in B_{1\dots q-1}$  be, in  $G'$ , at distance at least  $\text{ecc}^*(v, B_{1\dots q-1})$  to any vertex of  $B_q$  (including  $v$ ) —  $x$  exists by Claim 5. Then,  $x \in X$ .*

**Proof:** Towards a contradiction, assume  $x \notin X$ . Then, the distance of  $x$  and  $x_1$  in  $G'$  is at least  $\text{ecc}^*(v, B_{1\dots q-1}) - 1$ . By Claim 5, there is a vertex  $y \in B_1$  (thus  $y \in X$ ) that is, in  $G'$ , at distance at least  $\text{ecc}^*(v, B_1)$  to both  $v$  and  $x_1$  (as  $x_1 \in B_p \neq B_1$ ). As a shortest path between  $x$  and  $y$  in  $G'$  necessarily contains  $v$  or  $x_1$ , these vertices are at distance at least  $\text{ecc}^*(v, B_{1\dots q-1}) - 1 + \text{ecc}^*(v, B_1)$  to each other. This distance is more than  $2\text{ecc}^*(v, B_1) > D$ , a contradiction.  $\square$

**Claim 11**  $\text{ecc}^*(v, B_{1\dots q}) = \text{ecc}^*(v, B_{1\dots q-1})$ .

**Proof:** By Lemma 1, it suffices to show  $\text{ecc}^*(v, B_{1\dots q}) \leq \text{ecc}^*(v, B_{1\dots q-1})$ . To this end, let  $H$  be the outerplanar completion of  $B_{1\dots q}$  obtained from  $G'$  by contracting every branch  $B_i$ , with  $i > q$ , onto  $v$ . Since  $H$  is a minor of  $G'$ ,  $H$  is a diameter- $D$  outerplanar completion of  $B_{1\dots q}$ . We show  $\text{ecc}(v, H) \leq \text{ecc}^*(v, B_{1\dots q-1})$ .

Let  $x$  be a vertex that is, in  $G'$  and in  $H$ , at distance  $\text{ecc}^*(v, B_{1\dots q-1})$  to  $v$ , and at distance at least  $\text{ecc}^*(v, B_{1\dots q-1}) - 1$  to  $x_1$ . By Claim 10,  $x \in X$ . As  $H$  has diameter at most  $D$ , any vertex out of  $X$  is either at distance at most  $D - \text{ecc}^*(v, B_{1\dots q-1})$  to  $v$ , or at distance at most  $D + 1 - \text{ecc}^*(v, B_{1\dots q-1})$  to  $x_1$  (and thus at distance at most  $D + 2 - \text{ecc}^*(v, B_{1\dots q-1})$  to  $v$ ). As  $\text{ecc}^*(v, B_{1\dots q-1}) > D/2 + 1$ , any vertex out of  $X$  is thus at distance at most  $\text{ecc}^*(v, B_{1\dots q-1})$  from  $v$ . Finally Claim 9 implies that the vertices of  $X$  are also at distance at most  $\text{ecc}^*(v, B_{1\dots q-1})$  from  $v$  in  $H$ .  $\square$

We now claim that there exist two consecutive such values  $q$  between 4 and  $s \geq 7$ . Indeed, note first that  $B_1$  is linked to at most two other branches, as otherwise these branches together with  $v$  and  $B_1$  would induce a  $K_{2,3}$ -minor. Note also that by Claim 4 and Claim 6,  $B_2$  is linked to  $B_1$ , so it follows that  $B_1$  is linked to at most one branch  $B_j$  with  $j \geq 3$ . Therefore, for  $4 \leq q \leq 7$ , there are at least two consecutive values of  $q$  such that  $B_q$  is not linked to  $B_1$ , as we wanted to prove.



Once we have two consecutive such values  $q$  between 4 and  $s \geq 7$ , we have by Claim 11 that  $\text{ecc}^*(v, B_{1\dots i}) = \text{ecc}^*(v, B_{1\dots i+2})$ , for some  $i \leq s - 2$ , contradicting Claim 8. This concludes the proof of the lemma.  $\square$

Our algorithm will compute the eccentricity of a given “root” vertex in a diameter- $D$  outerplanar completion  $G'$  of  $G$  in which this eccentricity is minimal, that is,  $\text{ecc}^*(r, G)$ . Then, however, the branch containing the root ( $B_0$  in Algorithm 1, Subsection 2.3) should not be removed. Therefor, although Lemma 2 already implies that  $G$  has a diameter- $D$  outerplanar completion if and only if  $G[B_{1\dots 7}]$  does, we instead use the following corollary to identify removable branches.

**Corollary 1** *Let  $G$  be an outerplanar graph with a cut vertex  $v$  that belongs to at least 8 branches. Denote these branches  $B_1, \dots, B_t$ , with  $t \geq 8$ , in such a way that  $\text{ecc}^*(v, B_1) \geq \text{ecc}^*(v, B_2) \geq \dots \geq \text{ecc}^*(v, B_t)$ . For each  $8 \leq i \leq t$ , the graph  $G_i = \bigcup_{j \in \{1, \dots, 7, i\}} B_j$  has a diameter- $D$  outerplanar completion if and only if  $G$  does.*

**Proof:** Recall that the property of having an outerplanar completion with bounded diameter is minor closed. Thus  $G_i$  being a minor of  $G$ , we have that if  $G$  admits a diameter- $D$  outerplanar completion so does  $G_i$ .

On the other hand, if  $G_i$  admits a diameter- $D$  outerplanar completion, by Lemma 2 applied to  $G_i$  we have that  $\text{ecc}^*(v, B_{1\dots 6}) + \text{ecc}^*(v, B_7) \leq D$ . Thus gluing on  $v$  the outerplanar completions of  $G[B_{1\dots 6}], G[B_7], \dots, G[B_t]$ , respectively achieving  $\text{ecc}^*(v, B_{1\dots 6}), \text{ecc}^*(v, B_7), \dots, \text{ecc}^*(v, B_t)$ , one obtains a diameter- $D$  outerplanar completion of  $G$ .  $\square$

## 2.2 Dealing with 2-vertex separators

In this subsection, we extend the definition of eccentricity to the pairs  $(u, v)$  such that  $uv \in E(G)$ . Namely,  $\text{ecc}(u, v, G)$  is defined as the set of pairs obtained by taking the maximal elements of the set  $\{(\text{dist}_G(u, w), \text{dist}_G(v, w)) \mid w \in V(G)\}$ . The pairs are ordered such that  $(d_1, d_2) \leq (d'_1, d'_2)$  if and only if  $d_1 \leq d'_1$  and  $d_2 \leq d'_2$ . As  $u$  and  $v$  are adjacent, note that  $\text{dist}_G(u, w)$  and  $\text{dist}_G(v, w)$  differ by at most one. Hence,  $\text{ecc}(u, v, G)$  equal to one of  $\{(d, d)\}$ ,  $\{(d, d+1)\}$ ,  $\{(d+1, d)\}$ , and  $\{(d, d+1), (d+1, d)\}$ , for some positive integer  $d$ .

**Lemma 3** *Consider a connected graph  $G$  with  $V(G) =: X$  and a triangle  $uvw$  and two sets  $X_u, X_v \subseteq X$  such that  $X_u \cup X_v = X$ ,  $X_u \cap X_v = \{w\}$ ,  $\partial(X_u) \subseteq \{u, w\}$ , and  $\partial(X_v) \subseteq \{v, w\}$ . Then  $\text{ecc}(u, v, G)$  equals the maximal elements of the set*

$$\{(d_u, \min\{d_u + 1, d_w + 1\}) \mid (d_u, d_w) \in \text{ecc}(u, w, X_u)\} \cup \{(\min\{d_w + 1, d_v + 1\}, d_v) \mid (d_w, d_v) \in \text{ecc}(w, v, X_v)\}.$$

**Proof:** It is clear from the fact that a shortest path from  $X_u \setminus \{u\}$  to  $u$  does not go through  $X_v \setminus \{w\}$  (as it should go through  $w \in N(u)$ ), from the fact that a shortest path from  $X_u$  to  $v$  goes through  $\{u, w\} \subseteq N(v)$ , and from the fact that any subpath of a shortest path is a shortest path (for some pair of vertices).  $\square$

Given a connected outerplanar graph  $G$ , for any two vertices  $u, v \in V(G)$  and any vertex set  $X \subseteq V(G)$  with  $u, v \in X$  such that  $\partial(X) \subseteq \{u, v\}$ , let us define  $\text{ecc}_D^*(u, v, X)$  as the minimal elements of the set

$$\left\{ \text{ecc}(u, v, H) \mid \begin{array}{l} H \text{ is a diameter-} D \text{ outerplanar completion of } G[X] \text{ such} \\ \text{that } uv \in E(H) \text{ and such that } uv \text{ lies on the outer face.} \end{array} \right\}$$

If this set is empty, we set  $\text{ecc}_D^*(u, v, X)$  to  $(\infty, \infty)$ . Similarly to Subsection 2.1, we may drop the subindex  $D$  from the notation  $\text{ecc}_D^*(u, v, X)$ . Here,  $\text{ecc}(u, v, H) \leq \text{ecc}(u, v, H')$  if and only if for any  $(d_1, d_2) \in \text{ecc}(u, v, H)$  there exists a  $(d'_1, d'_2) \in \text{ecc}(u, v, H')$  such that  $(d_1, d_2) \leq (d'_1, d'_2)$ . According to the possible forms of  $\text{ecc}(u, v, H)$ , we have that  $\text{ecc}^*(u, v, X)$  is of one of the following five forms:

- $\{(d, d)\}$ ,
- $\{(d, d+1)\}$ ,
- $\{(d+1, d)\}$ ,
- $\{(d, d+1), (d+1, d)\}$ , or
- $\{(d, d+1)\}, \{(d+1, d)\}$ ,

for some positive integer  $d$ .

Considering  $\text{ecc}^*(u, X)$  for some  $u$  and  $X$ , note that  $u$  has at least one incident edge  $uv$  on the outer face in the outerplanar completion achieving  $\text{ecc}^*(u, X)$ . Thus, we can observe the following.

**Observation 1**  $\text{ecc}^*(u, X) = \min_{v \in X} \min_{S \in \text{ecc}^*(u, v, X)} \max_{(d_u, d_v) \in S} d_u$ .

### 2.3 The algorithm for connected outerplanar graphs

We now proceed to describe a polynomial-time algorithm that solves OUTERPLANAR DIAMETER IMPROVEMENT when the input outerplanar graph is assumed to be connected. In Subsection 2.4 we will deal with the disconnected case. Before proceeding to the formal description of the algorithm, let us provide a high-level sketch.

Algorithm 1 described below receives a *connected* outerplanar graph  $G$ , an arbitrary non-cut vertex  $r$  of  $G$ , called the *root* (such a vertex is easily seen to exist in any graph), and a positive integer  $D$ . In order to decide whether  $G$  admits a diameter- $D$  outerplanar completion, we will compute in polynomial time the value of  $\text{ecc}_D^*(r, G)$ , which, by definition, is distinct from  $\infty$  if and only if  $G$  admits a diameter- $D$  outerplanar completion.

In order to compute  $\text{ecc}_D^*(r, G)$ , the algorithm proceeds as follows. In the first step (lines 1-9), we consider an arbitrary block  $B$  of  $G$  containing  $r$  (line 1), and in order to reduce the input graph  $G$ , we consider all cut vertices  $v$  of  $G$  in  $B$ . For each such cut vertex  $v$ , we order its corresponding branches according to their eccentricity w.r.t.  $v$  (line 8), and by Corollary 1 it is safe to keep just a constant number of them, namely 8 (line 9). For computing the eccentricity of the branches not containing the root (lines 5-7), the algorithm calls itself recursively, by considering the branch as input graph, and vertex  $v$  as the new root.

In the second step of the algorithm (lines 10-17), we try all 2-vertex separators  $u, v$  in the eventual completed graph  $G'$  (note that  $G'$  cannot be 3-connected, as otherwise it would contain a  $K_{2,3}$ -minor), together with a set  $X$  consisting of a subset of the connected components of  $G' \setminus \{u, v\}$ , not containing the root  $r$ . For each such triple  $(u, v, X)$ , our objective is to compute the value of  $\text{ecc}_D^*(u, v, X)$ . For doing so, after initializing its value (lines 11-12), we consider all possible triples  $w, X_u, X_v$  chosen as in Lemma 3 after adding the triangle  $uvw$  to  $G[X]$  (line 13), for which we already know the values of  $\text{ecc}_D^*(u, w, X_u)$  and  $\text{ecc}_D^*(w, v, X_v)$ , since the sets  $X$  are processed by increasing size. Among all choices of one element in  $\text{ecc}_D^*(u, w, X_u)$  and another in  $\text{ecc}_D^*(w, v, X_v)$  (line 14), only those whose corresponding completion achieves diameter at most  $D$  are considered for updating the value of  $\text{ecc}_D^*(u, v, X)$  (line 15). For updating  $\text{ecc}_D^*(u, v, X)$  (line 17), we first compute  $\text{ecc}_D(u, v, X)$  using Lemma 3 (line 16).

Finally, once we have computed all values of  $\text{ecc}_D^*(u, v, X)$ , we can easily compute the value of  $\text{ecc}_D^*(u, X)$  by using Observation 1 (line 18). We can now provide a formal description of the algorithm.

---

**Algorithm 1: OPDI-Connected**

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**Input** : A connected outerplanar graph  $G$ , a root  $r \in V(G)$  such that  $G \setminus \{r\}$  is connected, and a positive integer  $D$ .

**Output**:  $\text{ecc}_D^*(r, G)$ .

// all over the recursive calls of the algorithm,  $G$  is a global variable, which gets updated whenever some vertices are removed in line 9

- 1 Let  $B$  be a block of  $G$  containing  $r$
- // we consider all cut vertices of  $B$  and we reduce  $G$
- 2 **foreach** cut vertex  $v \in V(B)$  **do**
- 3   Let  $C_0, \dots, C_t$  be the connected components of  $G \setminus \{v\}$ , where  $r \in C_0$
- 4   Let  $B_0 \leftarrow C_0$  // the branch containing the root
- 5   **for**  $i \leftarrow 1$  **to**  $t$  **do**
- 6     Let  $B_i \leftarrow G[C_i \cup \{v\}]$  // the branches around  $v$
- 7      $\text{ecc}_i \leftarrow \text{OPDI-Connected}(B_i, v, D)$ . // recursive call to compute
- $\text{ecc}_D^*(v, B_i)$
- 8   Reorder the  $B_i$ 's so that  $\text{ecc}_1 \geq \text{ecc}_2 \geq \dots \geq \text{ecc}_t$
- 9   Remove  $B_8, \dots, B_t$  from  $G$  // by Corollary 1
- // guess all 2-separators  $u, v$  in the target completion  $G'$ , together with a subset  $X$  of the connected components of  $G' \setminus \{u, v\}$
- 10 **foreach** triple  $(u, v, X)$  such that  $r \notin X \setminus \{u, v\}$  and  $\partial(X) \subseteq \{u, v\}$  **do**
- // by increasing size of  $X$ , and only if the triple  $(u, v, X)$  has not already been considered before in a previous iteration
- 11    $\text{Tab}_{\text{ECC}}(u, v, X) \leftarrow \{(\infty, \infty)\}$ . // it corresponds to  $\text{ecc}_D^*(u, v, X)$
- 12   **if**  $X = \{u, v\}$  **then**  $\text{Tab}_{\text{ECC}}(u, v, X) \leftarrow \{(0, 1), (1, 0)\}$ .
- 13   **else foreach**  $w, X_u, X_v$  satisfying the hypothesis of Lemma 3 in the graph obtained from  $G[X]$  by adding the triangle  $uvw$  **do**
- // eccentricities of smaller subgraphs have been already computed
- 14     **foreach**  $S_u \in \text{Tab}_{\text{ECC}}(u, w, X_u)$  and  $S_v \in \text{Tab}_{\text{ECC}}(w, v, X_v)$  **do**
- 15       **if for all**  $(d_u, d_w^u) \in S_u$  and  $(d_v, d_w^v) \in S_v$ , we have  $(d_w^u + d_w^v \leq D)$  or  $(d_u + 1 + d_v \leq D)$  **then**
- // if the diameter of the considered completion of  $X$  is  $\leq D$ , we compute  $\text{ecc}_D(u, v, X)$  using Lemma 3
- 16           $\text{Ecc} \leftarrow \max \left\{ \{(d_u, \min\{d_u + 1, d_w + 1\}) \mid (d_u, d_w) \in S_u\} \cup \{(\min\{d_v + 1, d_w + 1\}, d_v) \mid (d_w, d_v) \in S_v\} \right\}$
- // update  $\text{ecc}_D^*(u, v, X)$
- 17           $\text{Tab}_{\text{ECC}}(u, v, X) \leftarrow \min\{\text{Tab}_{\text{ECC}}(u, v, X) \cup \text{Ecc}\}$
- // finally, we compute  $\text{ecc}_D^*(r, G)$  using Observation 1
- 18 **return**  $\min_{v \in V(G) \setminus \{r\}} \min_{S \in \text{Tab}_{\text{ECC}}(r, v, V(G))} \max_{(d_u, d_v) \in S} d_u$ .

---

The correctness of Algorithm 1 follows from the results proved in Subsections 2.1 and 2.2, and the following fact (whose proof is straightforward), which guarantees that the value of  $\text{ecc}_D^*(u, v, X)$  can indeed be computed as done in lines 13-17.

**Fact 1** *There exists an outerplanar completion  $H$  of  $G[X]$  with the edge  $uv$  on the outerboundary if and only if there is  $w \in X$  and two sets  $X_u, X_v$  such that:*

- (a)  $X_u \cup X_v = X$ ,  $X_u \cap X_v = \{w\}$ ,
- (b)  $\partial_G(X_u) \subseteq \{u, w\}$  and  $\partial_G(X_v) \subseteq \{v, w\}$ , and
- (c) *there exists an outerplanar completion  $H_u$  of  $G[X_u]$  with the edge  $uw$  on the outerboundary, and an outerplanar completion  $H_v$  of  $G[X_v]$  with the edge  $vw$  on the outerboundary.*

It remains to analyze the running time of the algorithm.

**Running time analysis of Algorithm 1.** Note that at line 6 each  $B_i$  is recursively replaced by an equivalent (by Corollary 1) subgraph such that its cut vertices have at most 8 branches attached.

Let us first focus on the second step of the algorithm, that is, on lines 10-17. The algorithm considers in line 10 at most  $O(n^2)$  pairs  $\{u, v\}$ . As each of  $u$  and  $v$  has at most 7 attached branches avoiding the root, and  $G \setminus \{u, v\}$  has at most 2 connected components with vertices adjacent to both  $u$  and  $v$  (as otherwise  $G$  would contain a  $K_{2,3}$ -minor), there are at most  $2^7 \cdot 2^7 \cdot 2^2 = 2^{16}$  possible choices for assigning these branches or components to  $X$  or not. In line 13, the algorithm considers  $O(n)$  vertices  $w$ . Similarly, as  $w$  belongs to at most 7 branches not containing  $u$  nor  $v$ , there are at most  $2^7$  choices for assigning these branches to  $X_u$  or  $X_v$ . In lines 14-17, the algorithm uses values that have been already computed in previous iterations, as the sets  $X$  are considered by increasing order. Note that each of  $\text{ecc}_D^*(u, w, X_u)$  and  $\text{ecc}_D^*(w, v, X_v)$  contains at most 2 elements, so at most 4 choices are considered in line 14. Again, at most 4 choices are considered in line 15. Therefore, lines 14-17 are executed in constant time.

As for the first step of the algorithm (lines 1-9), the algorithm calls itself recursively. The number of recursive calls is bounded by the number of blocks of  $G$ , as by construction of the algorithm each block is assigned a single root. Therefore, the number of recursive calls is  $O(n)$ . Once the algorithm calls itself and the corresponding branch has no cut vertex other than the root, the algorithm enters in lines 10-17, whose time complexity has already been accounted above. (Note that each triple  $(u, v, X)$  is considered only once, and the value of  $\text{ecc}_D^*(u, v, X)$  is stored in the tables.)

Finally, in line 18, the algorithm considers  $O(n)$  vertices, and for each of them it chooses among constantly many numbers. Summarizing, we have that the algorithm has overall complexity  $O(n^3)$ .

It is worth mentioning that Algorithm 1 can also compute the actual completion achieving diameter at most  $D$ , if any, within the same time bound. Indeed, it suffices to keep track of which edges have been added to  $G$  when considering the guessed triangles  $uvw$  (recall that we may assume that the completed graph is triangulated).

**Theorem 1** *Algorithm 1 solves OUTERPLANAR DIAMETER IMPROVEMENT for connected input graphs in time  $O(n^3)$ .*

Note that we can compute  $\text{opdi}(G)$  by calling Algorithm 1 with an arbitrary root  $r \in V(G)$  (such that  $G \setminus \{r\}$  is connected) for increasing values of  $D$ .

**Corollary 2** *Let  $G$  be a connected outerplanar graph. Then,  $\text{opdi}(G)$  can be computed in time  $O(n^4)$ .*

## 2.4 The algorithm for disconnected outerplanar graphs

In this subsection we will focus on the case where the input outerplanar graph is disconnected. The *radius* of a graph is defined as the eccentricity of a “central” vertex, that is, the minimum eccentricity of any of its vertices.

**Lemma 4 ([8], Theorem 3)** *Let  $G$  be a maximal outerplanar graph of diameter  $D$  and radius  $r$ . Then,  $r \leq \lfloor D/2 \rfloor + 1$ .*

In the following, we denote the minimum radius of a diameter- $D$  outerplanar completion of a graph or connected component  $G$  by  $r^*(G)$ . If  $G$  has no diameter- $D$  outerplanar completion, then let  $r^*(G) = \infty$ .

**Definition 1** *Let  $G$  be a connected graph and let  $D$  be an integer. Let  $G'$  be the graph resulting from  $G$  by adding an isolated vertex  $v$ . Let  $G^*$  be a diameter- $D$  outerplanar completion of  $G'$  that minimizes the eccentricity of  $v$ . Then,  $G^*$  is called *escalated completion of  $(G, D)$  with respect to  $v$*  and the eccentricity  $\text{ecc}(v, G^*)$ , denoted by  $r^+(G)$ , is called *escalated eccentricity of  $(G, D)$* . Again, if such a  $G^*$  does not exist, let  $r^+(G) = \infty$ .*

We will apply Definition 1 also to connected components of a graph and, if clear from context, we omit  $D$ . Note that we can compute  $r^+(G)$  by guessing an edge between the isolated vertex  $v$  and  $G$  and running **OPDI-Connected**, the algorithm for connected graphs. Hence this can be done in  $O(n^4)$  time. Also note that  $r^*(G) \leq r^+(G) \leq r^*(G) + 1$ .

**Lemma 5** *Given a graph  $G$  with a connected component  $C$  such that  $r^+(C) < D/2$ , then  $G$  has a diameter- $D$  outerplanar completion if and only if  $G \setminus C$  does.*

**Proof:** In a diameter- $D$  outerplanar completion of  $G \setminus C$  there is a vertex  $v$  with eccentricity at most  $\lfloor D/2 \rfloor + 1$ , by Lemma 4. In this completion, adding the completion of  $C + v$  achieving  $r^+(C) < D/2$ , yields a diameter- $D$  outerplanar completion of  $G$ .  $\square$

**Observation 2** *Let  $C$  be a connected component of  $G$ , let  $G'$  be an outerplanar completion of  $G$  and let  $C'$  be a connected component of  $G' \setminus C$ . Then, there is a vertex  $v \in C$  at distance at least  $r^+(C)$  to each vertex of  $C'$  in  $G'$ .*

**Proof:** Let the result of contracting all vertices in  $G' \setminus (C \cup C')$  onto vertices in  $C$  and contracting  $C'$  onto a single vertex  $u$  be  $G''$ . Then,  $G''$  is a subgraph of an outerplanar completion of the result of adding  $u$  as isolated vertex to  $G'[C]$ . By definition,  $\text{ecc}(u, G'') \geq r^+(C)$ , implying that there is a vertex  $v \in C$  at distance at least  $r^+(C)$  to  $u$  in  $G''$ . Thus,  $v$  is at distance at least  $r^+(C)$  to each vertex of  $C'$  in  $G'$ .  $\square$

Observation 2 immediately implies that any cutset separating two connected components  $C_1$  and  $C_2$  of  $G$  in  $G'$  has distance at least  $r^+(C_1)$  and  $r^+(C_2)$  to some vertex in  $C_1$  and  $C_2$ , respectively. Thus, these two vertices are at distance at least  $r^+(C_1) + r^+(C_2)$  in  $G'$ .

**Corollary 3** *Let  $C_1$  and  $C_2$  be connected components of  $G$  such that  $r^+(C_1) + r^+(C_2) > D$  and let  $G'$  be a diameter- $D$  outerplanar completion of  $G$ . Then,  $C_1$  and  $C_2$  are adjacent in  $G'$ , i.e.  $G'$  has an edge with an end in  $C_1$  and an end in  $C_2$ .*

Corollary 3 allows us to conclude that all connected components  $C$  with  $r^+(C) > D/2$  have to be pairwise adjacent in any diameter- $D$  outerplanar completion of  $G$ . Thus, there cannot be more than three such components.

**Lemma 6** *An outerplanar graph  $G$  with more than 3 connected components  $C$  such that  $r^+(C) > D/2$  has no diameter- $D$  outerplanar completion. On the other hand, if  $G$  has no connected component  $C$  such that  $r^+(C) > D/2$ , then  $G$  necessarily has a diameter- $D$  outerplanar completion.*

**Proof:** The first statement comes from the above comments. The proof of the second statement is similar to the one of Lemma 5. For some component  $C$  of  $G$ , let  $v$  be such that  $\text{ecc}(v, C) = r^*(C) \leq r^+(C) \leq D/2$ , and complete  $C$  in order to achieve this value. Then for the other components  $C'$  consider their escalated completion with respect to  $v$ . As  $r^+(C') \leq D/2$  this graph has diameter at most  $D$ .  $\square$

Hence, assume  $G$  has  $p = 1, 2$ , or 3 connected components  $C$  such that  $r^+(C) > D/2$ . By Corollary 3 these  $p$  components are pairwise adjacent in the desired completion. Note that with  $O(n^{2p-2})$  tries, we can guess  $p - 1$  edges connecting all such components into one larger component. Thus, in the following, we assume that there is only one component  $C$  with  $r^+(C) > D/2$ , denoted  $C_{\max}$ .

**Lemma 7** *Consider an outerplanar graph  $G$  with exactly one connected components  $C_{\max}$  such that  $D/2 < r^+(C_{\max}) < \infty$ . If  $r^*(C_{\max}) \leq D/2$ , then  $G$  necessarily has a diameter- $D$  outerplanar completion.*

**Proof:** Same proof as Lemma 6.  $\square$

Let us now distinguish two cases according to the parity of  $D$ .

**Lemma 8** *For odd  $D$ , if an outerplanar graph  $G$  has at most one component  $C_{\max}$  such that  $D/2 < r^+(C_{\max}) < \infty$ , then  $G$  has a diameter- $D$  outerplanar completion.*

**Proof:** Indeed, by Lemma 5 it is sufficient to consider the component  $C_{\max}$  alone. As  $r^+(C_{\max}) < \infty$ ,  $C_{\max}$  has a diameter- $D$  outerplanar completion, and so does  $G$ .  $\square$

The case where  $D$  is even is more technical.

**Lemma 9** *For even  $D$ , Let  $p$  and  $q$  respectively denote the number of connected components  $C$  such that  $D/2 < r^+(C) < \infty$  and  $r^+(C) = D/2$ , of an outerplanar graph  $G$ . If  $p \geq 2$  and  $p + q \geq 5$ , then  $G$  has no diameter- $D$  outerplanar completion.*

**Proof:** By Corollary 3, in a diameter- $D$  outerplanar completion  $G'$  of  $G$  the  $p$  components are pairwise adjacent, and any of the  $q$  components is adjacent to every of the  $p$  ones. For  $p = 2$ , as  $q \geq 3$ , this would induce a  $K_{2,3}$ -minor in  $G'$ , a contradiction. For the other cases, this would induce a  $K_4$ -minor in  $G'$ , a contradiction.  $\square$

**Lemma 10** *For even  $D$ , if an outerplanar graph  $G$  has one component, denoted  $C_{\max}$ , such that  $D/2 < r^*(C_{\max}) < \infty$ , and at least 4 other components  $C$  such that  $D/2 \leq r^+(C) < \infty$ , then  $G$  has no diameter- $D$  outerplanar completion.*

**Proof:** Let us denote  $C_1, C_2, C_3$ , and  $C_4$  the connected components such that  $r^+(C_i) \geq D/2$ , distinct from  $C_{\max}$ . Assume for contradiction that  $G$  admits a diameter- $D$  outerplanar completion, denoted  $G'$ .

**Claim 12** *For each  $C_i, C_j$ , either  $C_i$  and  $C_j$  are adjacent in  $G'$ , or  $C_i$  and  $C_j$  have a common neighbor in  $G'$ .*

**Proof:** Assume for contradiction that  $C_i$  and  $C_j$  are not adjacent and do not have a common neighbor in  $G'$ . Let us now construct the graph  $G''$  from  $G'$  as follows. For any component  $C$  of  $G' \setminus (C_i \cup C_j)$  that is not adjacent to both  $C_i$  and  $C_j$ , contract  $C$  onto vertices of  $C_i$  or  $C_j$  (According to the one  $C$  is neighboring). As  $G''$  is obtained from  $G'$  by contracting edges,  $G''$  also is a diameter- $D$  outerplanar completion (for some graph containing  $C_i$  and  $C_j$ ). Let  $N_i := N_{G''}(C_i)$ , let  $N_j := N_{G''}(C_j)$ , and note that  $C_i \cap N_j = \emptyset$ ,  $N_i \cap C_j = \emptyset$ , and  $N_i \cap N_j = \emptyset$ . Then, by Observation 2 (as  $G'' \setminus C_i$  and  $G'' \setminus C_j$  are connected), there are vertices  $v_i \in C_i$  and  $v_j \in C_j$  at distance at least  $D/2$  to each vertex in  $N_i$  and  $N_j$ , respectively, in  $G''$ . Since  $N_i$  and  $N_j$  are at distance at least one,  $v_i$  and  $v_j$  are at distance at least  $D+1$ , contradicting  $G''$  having diameter  $D$ .  $\square$

**Claim 13** *There is a vertex  $u \in \mathcal{C}_{\max}$  that is adjacent in  $G'$  to 3 of the components  $C_1, C_2, C_3$ , and  $C_4$ .*

**Proof:** First, note that there is a vertex  $u$  and 3 components, say  $C_1, C_2, C_3$ , with  $u \in N_{G'}[C_i]$  for all  $1 \leq i \leq 3$ , since otherwise, there would be internally vertex-disjoint paths between each two of the four components  $C_i$ , implying the existence of a  $K_4$ -minor in  $G'$ .

If  $u$  is neither in  $\mathcal{C}_{\max}$  nor in  $C_i$ , for  $1 \leq i \leq 3$ , then, since all the  $C_i$  are adjacent to  $\mathcal{C}_{\max}$  (by Corollary 3),  $G'$  would have a  $K_{2,3}$ -minor on the vertex sets  $\{u, \mathcal{C}_{\max}\}$  and  $\{C_1, C_2, C_3\}$ .

Hence, in the following, we assume that  $u \in C_1$ . Let  $z$  be a neighbor of  $C_1$  in  $\mathcal{C}_{\max}$  and, for  $i \in \{2, 3\}$  let  $w_i$  denote a neighbor of  $C_4$  in  $N[C_i]$ . We note that  $w_2 \neq z$  and  $w_3 \neq z$ , since otherwise, the claim follows and we are done. Furthermore,  $w_2 \neq u$  and  $w_3 \neq u$ , since otherwise there is a  $K_{2,3}$ -minor on the vertex sets  $\{u, \mathcal{C}_{\max}\}$  and  $\{C_2, C_3, C_4\}$ . Let  $X := (C_4 \cup \{w_2, w_3\}) \setminus (C_2 \cup C_3)$  and note that  $X$  is adjacent to  $C_2$  and  $C_3$ , respectively. Let  $Y$  be the connected component of  $\mathcal{C}_{\max} \setminus \{w_2, w_3\}$  containing  $z$ , and note that  $Y$  is adjacent to  $C_1$  and  $X$ . Finally, since  $X, Y, C_1, C_2$ , and  $C_3$  are pairwise disjoint,  $G'$  has a  $K_{2,3}$ -minor on the vertex sets  $\{X, C_1\}$  and  $\{C_2, C_3, Y\}$ .  $\square$

Let  $v$  denote a vertex of  $\mathcal{C}_{\max}$  that is at distance at least  $D/2 + 1$  to  $u$  in  $G'$  and consider the result  $G' \setminus \{u\}$  of removing  $u$  from  $G'$ . Let  $C$  denote the connected component of  $G' \setminus \{u\}$  that contains  $v$ . Towards a contradiction, assume there is a connected component  $C_i$  that is adjacent to  $u$  but not to  $C$  in  $G'$ , then all paths between  $v$  and any vertex in  $C_i$  contain  $u$ . Since  $G'$  has diameter  $D$ , all vertices in  $C_i$  are at distance at most  $D/2 - 1$  to  $u$  in  $G'$ , contradicting  $r^+(C_i) \geq D/2$ . Thus there is a  $K_{2,3}$ -minor in  $G'$  on the vertex sets  $\{C_1, C_2, C_3\}$  and  $\{u, X\}$  where  $X$  is the connected component of  $G' \setminus (C_1 \cup C_2 \cup C_3 \cup \{u\})$  containing  $v$ . This concludes the proof of the lemma.  $\square$

Hence, assume  $G$  has  $q = 0, 1, 2$ , or 3 connected components  $C$  such that  $r^+(C) = D/2$ . By Corollary 3 these  $q$  components are adjacent to each of the  $p$  components such that  $r^+(C) > D/2$ . Note that with  $O(n^{2q})$  tries, we can guess  $q$  edges connecting each of the  $q$  components to one of the  $p$  component. Then we are left with a connected graph, and we can call **OPDI-Connected**.

**The algorithm itself.** We now describe a polynomial-time algorithm that solves the OUTERPLANAR DIAMETER IMPROVEMENT problem when the input contains a disconnected outerplanar graph. Algorithm 2 described below receives a (disconnected) outerplanar graph  $G$ , and a positive integer  $D$ .

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**Algorithm 2:** OPDI-Disconnected

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**Input** : A disconnected outerplanar graph  $G = (V, E)$  and an integer  $D$ .  
**Output**: ‘TRUE’ if and only if  $G$  has a diameter- $D$  outerplanar completion.

```

1 foreach connected component  $C$  of  $G$  do
2    $r^+(C) \leftarrow \infty$ 
3    $r^*(C) \leftarrow \infty$ 
4   foreach  $u \in V(C)$  do
5      $Ecc \leftarrow \text{OPDI-Connected}(C, u, D)$ 
6      $r^*(C) \leftarrow \min\{r^*(C), Ecc\}$ 
7      $C' \leftarrow C$  with added vertex  $v$  and added edge  $uv$ 
8      $Ecc \leftarrow \text{OPDI-Connected}(C', v, D)$ 
9      $r^+(C) \leftarrow \min\{r^+(C), Ecc\}$ 
10  if  $r^+(C) = \infty$  then return FALSE
11  if  $r^+(C) < D/2$  then Remove  $C$  from  $G$ 
12 if  $r^+(C) \leq D/2$  for every  $C$  then return TRUE
13 if  $r^+(C) \leq D/2$  for every  $C$  except one,  $C_{\max}$ , and  $r^*(C_{\max}) \leq D/2$  then
14   return TRUE
15 if  $r^+(C) > D/2$  for at least 4 conn. components  $C$  then return FALSE
16 foreach choice of edges interconnecting these  $p = 1, 2$ , or 3 connected
    components do
17   // choose  $p - 1$  edges
18   Let  $C_{\max}$  be this new conn. component
19   if  $\text{OPDI-Connected}(C_{\max}, v, D) < \infty$  then
20     if  $D$  is odd then return TRUE if  $G$  has more than  $5 - p$  conn. comp.
21     then
22       //  $C_{\max}$  and  $q$  connected comp. such that  $r^+(C) = D/2$ 
23       return FALSE
24     else
25       foreach choice of  $q$  edges connecting  $G$  do
26         if  $\text{OPDI-Connected}(G, v, D) < \infty$  then return TRUE
27         return FALSE
28 return FALSE

```

---

At the beginning, the algorithm computes  $r^+(C)$  and  $r^*(C)$  for each connected component  $C$  of  $G$ . For computing  $r^+(C)$  the algorithm adds a vertex  $v$ , guessing (with  $O(n)$  tries) an edge connecting  $v$  to  $C$ , and then calls **OPDI-Connected** for this component and root  $v$ . For computing  $r^*(C)$  the algorithm guesses a root  $u$  (with  $O(n)$  tries), and then calls **OPDI-Connected** for  $C$  and root  $u$ .

If  $r^*(C) = \infty$  for some component  $C$  then, as  $r^*(G) \geq r^*(C)$ ,  $G$  has no diameter- $D$  outerplanar completion.

Then, as they could be added in a diameter- $D$  outerplanar completion (by Lemma 5), the algorithm removes the components  $C$  with small escalated eccentricity, that is those such that  $r^+(C) < D/2$ .



Then the algorithm tests if there is no component  $C$  such that  $r^+(C) > D/2$ , or if there is only one component  $C$  such that  $r^+(C) > D/2$ , and if  $r^*(C) \leq D/2$ . In both cases by Lemma 6 and Lemma 7,  $G$  is a positive instance.

Then the algorithm tests if there is more than 3 component  $C$  such that  $r^+(C) > D/2$ . In this cases, by Lemma 6,  $G$  is a negative instance. Otherwise,  $G$  has  $p = 1, 2$ , or  $3$  such connected components, and the algorithm guesses  $p - 1$  edges (in time  $O(n^{2p-2})$ ) to connect them (as they should be by Corollary 3). For each such graph we call algorithm **OPDI-Connected** to check that this graph has a diameter- $D$  outerplanar completion.

Then the algorithm proceeds differently according to  $D$ 's parity. If  $D$  is odd, then  $G$  is a positive instance (By Lemma 8). If  $D$  is even, if  $G$  has (still) more than  $5 - p$  connected components (by Lemma 9 and Lemma 10), then  $G$  is a negative instance. Then we are left with a graph  $G$  with  $1 + q$  connected components, and again the algorithm guesses  $q$  edges (in time  $O(n^{2q})$ ), connecting  $G$ . For each of these graphs the algorithm calls **OPDI-Connected**( $G, v, D$ ) (for any  $v$ ) to check whether this graph admits a diameter- $D$  outerplanar completion.

Finally if none of these “guessed” connected graphs has a diameter- $D$  outerplanar completion, then the algorithm concludes that  $G$  is a negative instance.

**Theorem 2** *Algorithm 2 solves OUTERPLANAR DIAMETER IMPROVEMENT for disconnected input graphs in polynomial time. For odd  $D$  the running time is  $O(n^7)$ , while it is  $O(n^9)$  for even  $D$ .*

**Proof:** Indeed, the algorithm runs in time  $O(n^7)$  for odd  $D$  (at most  $O(n^4)$  at line 16, times  $O(n^3)$  for the call to **OPDI-Connected** in line 18). The algorithm runs in  $O(n^{2p+2q+1})$  time for even  $D$  ( $O(n^{2p-2})$  in line 16, times  $O(n^{2q})$  in line 22, times  $O(n^3)$  for the call to **OPDI-Connected** in line 23), where  $p$  and  $q$  respectively denote the number of connected components  $C$  such that  $r^+(C) > D/2$  and  $r^+(C) = D/2$ . As  $p + q \leq 4$ , we are done.  $\square$

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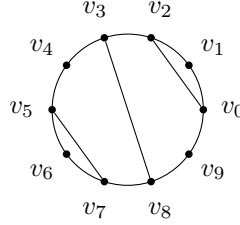
## A A partial list of forbidden substructures

In this section we provide a partial list of forbidden substructures that cannot appear as minors in graphs  $G$  such that  $\mathbf{opdi}(G) \leq D$ , for any integer positive  $D$ . We hope that these constructions will turn useful towards obtaining a complete (finite) obstruction set. Our main tool is the notion of parallel matching, which is strongly related to the diameter of a maximal outerplanar graph (Subsection A.1). We then present the forbidden substructures in Subsection A.2.

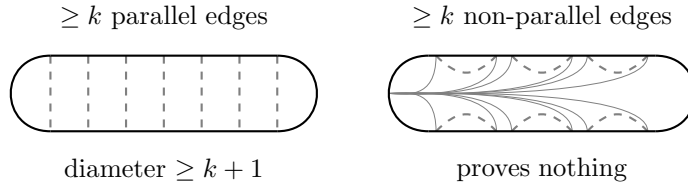
### A.1 Maximum parallel matching and diameter

We start by defining the notion of parallel matching.

**Parallel matching.** The diameter of an outerplanar graph  $G$  can be visualized by a specific kind of matching. Indeed, by defining an embedding of an outerplanar graph as a cyclic ordering of its vertex set around the outerboundary, we say that a matching  $u_1v_1, \dots, u_kv_k$  is a collection of *parallel edges* (or a *parallel matching*) if  $u_1, \dots, u_k, v_k, \dots, v_1$  is cyclically ordered with respect to the embedding. We further require that some vertex should exist between  $u_1$  and  $v_1$ , as well as between  $u_k$  and  $v_k$ .



Consequently, a sequence  $u_1v_1, \dots, u_kv_k \in E(G)$  of parallel edges can be seen as a laminar system of 2-cuts, i.e., in every  $G \setminus \{u_i, v_i\}$  the sets  $\{u_1v_1, \dots, u_{i-1}v_{i-1}\}$  and  $\{u_{i+1}v_{i+1}, \dots, u_kv_k\}$  belong to two distinct connected components<sup>2</sup>. In particular, each pair of vertices –including the first and the last one– must be a 2-cut of the graph.



**The maximum parallel matching is smaller than the diameter.** Because each pair of vertices in a parallel matching of size  $k$  is a 2-cut of the outerplanar graph and because they form a laminar collection of cuts, we can ensure that at least two vertices are at distance at least  $k + 1$ . Indeed, the first edge  $u_1v_1$  of the parallel matching cuts the graph into at least two connected components, one of which does not contain any edge of the matching. In such a connected component we take an arbitrary vertex  $s$ , and obtain another vertex  $t$  with the same procedure applied to edge  $u_kv_k$ . The distance  $\text{dist}(s, t)$  is necessarily larger than  $k + 1$ .

<sup>2</sup>Note that this can also be taken as a definition of parallel matchings independent of the embedding, in the specific case where  $G$  is maximal.



**Observation 3** *Let  $G$  be an outerplanar graph of diameter  $D$  and let  $M$  be a parallel matching in  $G$ . Then,  $|M| < D$ .*

**They are not equal in general.** Of course, there may very well exist outerplanar graphs with a very small maximum parallel matching, and an arbitrary large diameter. Cycles and paths are the easiest such examples, as their natural embedding contain no non-empty parallel matching, despite having unbounded diameter.

**They are equivalent for maximal outerplanar graphs.** In the specific case of maximal outerplanar graphs (whose embedding is unique), the size of the maximum parallel matching and the diameter differ by exactly one. We need the following lemma.

**Lemma 11** *Let  $G$  be a chordal graph and  $s, t \in V(G)$ . There exist  $\text{dist}(s, t) - 1$  vertex-disjoint cliques, each of which is an  $s, t$ -cut in  $G$ .*

**Proof:** First, let us notice that the lemma holds for  $G$  if and only if it holds for  $G \setminus v$ , where  $v \in V(G) \setminus \{s, t\}$  is a simplicial vertex. As a result, we can assume that either  $s$  or  $t$  is simplicial, and by symmetry that  $s$  is simplicial.

We can now consider the graph  $G'$  obtained from  $G$  by contracting the closed neighborhood  $N_G[s]$  onto a vertex  $s'$ . Since chordal graphs are closed under edge contraction,  $G'$  is chordal. Thus, we can apply the lemma recursively on the pair  $\{s', t\}$  of vertices with distance  $\text{dist}_{G'}(s', t) = \text{dist}_G(s', t) - 1$ . This yields a collection of  $\text{dist}(s, t) - 2$  cliques with –together with the clique  $N_G(s)$ – prove the lemma.  $\square$

In the particular case of a 2-connected outerplanar graph, these cliques are all of cardinality 2. Indeed, they are all obtained from the neighborhood of a simplicial vertex, and a simplicial vertex of an outerplanar graph has maximum degree 2. Furthermore, a laminar sequence of 2-cuts in a maximal outerplanar graph is necessarily a parallel matching, as noted above.

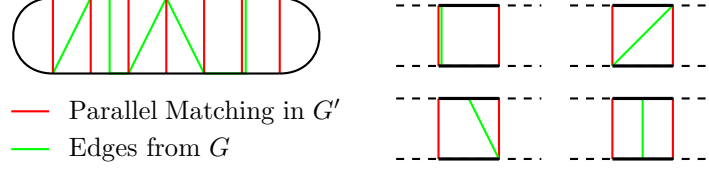
**Corollary 4** *Let  $G$  be a maximal outerplanar graph of diameter  $D$  and let  $M$  be a maximum parallel matching in  $G$ . Then,  $|M| = D - 1$ .*

**Edge-completion and maximum parallel matching.** An embedded outerplanar graph  $G$  with a maximum parallel matching of cardinality  $k - 1$  can be completed by addition of edges into an outerplanar graph of diameter at most  $2k$ . The procedure is quite simple: for each face of the outerplanar graph, add an edge from one vertex to all others and call the result  $G'$ . We will prove the bound on the diameter by showing that the size of a maximum parallel matching does not increase too much when the “star” edges are added.

Let us consider a maximum parallel matching of cardinality  $k'$  in  $G'$  (see the red edges in the picture below). We want to prove that, between any two consecutive edges of the matching of  $G'$ , there exists an edge from  $G$  which can replace them both in a parallel matching. We distinguish two cases:

- If one of the two consecutive (red) edges is also an edge of  $G$ , there is nothing to prove.
- If both of them are star edges, they belong to different faces of  $G$  (as they would otherwise share an endpoint). As a result, there is an edge of  $G$  which separates those two faces, and this is the edge we are looking for.

Note that if  $e_1, e_2, e_3$  are three consecutive (red) edges, the edge of  $G$  associated with the pair  $e_1, e_2$  may be the same as the edge associated with the pair  $e_2, e_3$ : this happens when  $e_2$  itself is an edge of  $G$ . This need not trouble us.



As the maximum parallel matching in  $G'$  has cardinality  $k'$ , there are  $k' - 1$  different pairs of consecutive (red) edges, each associated with a (green) edge from  $G$ . These (green) edges do not necessarily define a parallel matching by themselves, for they may touch at their endpoints or even be equal. Hence, we can obtain a matching in  $G$  by using only every second (green) edge associated with consecutive (red) edges, thus producing a parallel matching of length  $\lceil \frac{k'-1}{2} \rceil$ .

Consequently, the completion  $G'$  of  $G$  has diameter  $k' + 1$ , and the diameter of  $G$  itself is at least  $\lceil \frac{k'-1}{2} \rceil + 1$ . As

$$k' + 1 \leq 2 \left( \lceil \frac{k'-1}{2} \rceil + 1 \right),$$

the completion of  $G$  with stars is a 2-approximation of the completion of  $G$  achieving the smallest diameter.

## A.2 Forbidden substructures

In this subsection we present graphs  $G$  with  $\text{opdi}(G) > D$ . Because a supergraph  $H$  of such a graph  $G$  satisfies  $\text{opdi}(H) \geq \text{opdi}(G) > D$ , the following graphs can be seen as forbidden patterns for graphs which admit a diameter- $D$  outerplanar completion and we call them *diameter- $D$  obstruction*. The only diameter-1 obstruction is  $4K_1$ , and indeed any outerplanar graph on at least 4 vertices has diameter strictly greater than 1. We proceed to give a detailed description of forbidden subgraphs for larger diameters.

If  $X$  is a family of graphs, let  $iX$  denote the disjoint union of  $i$  graphs of  $X$ .  $\text{join}(v, iX)$  denotes the result of taking  $i$  graphs of  $X$  and the vertex  $v$  and connecting each of the  $i$  graphs to  $v$  by a single edge. We construct families  $A_i$  and  $B_i$  of diameter- $i$  obstructions:

$$A_i := \begin{cases} \{2K_1\} & \text{if } i = 0 \\ 2B_{i-1} & \text{if } 2|i \\ 2A_{i-1} & \text{if } 2 \nmid i \end{cases} \quad B_i := \begin{cases} \{K_{1,3}\} & \text{if } i = 1 \\ \emptyset & \text{if } 2|i \\ \text{join}(v, 3B_{i-2}) & \text{if } 2 \nmid i \end{cases}$$

Note that all graphs in  $B_i$  are connected.

**Observation 4** *Let  $G$  be a maximal graph and let  $G'$  be a connected induced subgraph of  $G$ . Then, there is a triangulation of  $G'$  that is also a contraction of  $G$ .*

**Definition 2** *Let  $G$  be an outerplane graph with parallel matching  $M = \{u_1v_1, u_2v_2, \dots, u_kv_k\}$  and let  $v$  be a vertex of  $G$  such that  $v$  is between  $u_k$  and  $v_k$  in the embedding of  $G$ . Then, we call  $M$  compatible with  $v$ .*

**Theorem 3** *Let  $D \in \mathbb{N}$  and let  $G$  be an outerplanar graph that contains a graph from  $A_D \cup B_D$  as a minor. Then  $G$  has diameter at least  $D + 1$ .*

**Proof:** Assume for contradiction that  $G$  has diameter at most  $D$ , and also that  $G$  is maximal (without loss of generality). The proof is by induction on  $D$ .

**Base:** For  $D = 0$ , note that  $A_0 \cup B_0 = \{2K_1\}$ . Consequently,  $G$  contains at least two vertices and has positive diameter. If  $D = 1$ , we have  $A_1 \cup B_1 = \{4K_1, K_{1,3}\}$ . Consequently,  $G$  contains at least four vertices and has diameter at least 2, given that  $K_4$  is not outerplanar.

**Inductive step:** Let  $D \geq 2$  and assume that the claim holds for all graphs of diameter strictly less than  $D$ .

*Case 1.  $2|D$ .* Then,  $A_D \cup B_D = 2B_{D-1}$ . Thus, there is a partition  $V_1 \uplus V_2$  of  $V(G)$  such that for each  $i \in \{1, 2\}$ ,  $G[V_i]$  is connected and contains some graph of  $B_{D-1}$  as a minor. By Observation 4, there is a triangulation  $G_1$  of  $G[V_1]$  which is a contraction of  $G$ . Let  $M_1 = \{u_1v_1, u_2v_2, \dots, u_kv_k\}$  be a maximum parallel matching in  $G_1$  and note that, by Observation 4, we know that  $k \geq \text{diam}(G_1) - 1$ . By induction hypothesis,  $k \geq D - 1$ . Without loss of generality, let  $\ell_1$  be such that  $G_2$  is embedded entirely between  $u_{\ell_1}$  and  $u_{\ell_1+1}$ . If  $\ell_1 \geq k/2$ , then let  $M'_1 := \{u_iv_i \mid 1 \leq i \leq k/2\}$ . Otherwise, let  $M'_1 := \{u_iv_i \mid k/2 \leq i \leq k\}$ . In both cases,  $M'_1$  is compatible with each vertex in  $V_2$  and  $|M'_1| \geq k/2 \geq (D-1)/2$ . Since  $2|D$ , we conclude  $|M'_1| \geq D/2$ . Likewise, there is a parallel matching  $M'_2$  in  $G_2$  that is compatible with each vertex in  $V_1$  and  $|M'_2| \geq D/2$ . Since  $V_1$  and  $V_2$  are disjoint,  $M_1 \cup M_2$  is a parallel matching in  $G$  and  $|M_1 \cup M_2| \geq D$ . Then, the claim follows by Observation 3.

*Case 2.  $2 \nmid D$ .* Then,  $G$  either contains a graph in  $2A_{D-1} = 4B_{D-2}$ , or a graph in  $\text{join}(v, 3B_{D-2})$  as a minor.

**Claim 14** *There are pairwise disjoint vertex sets  $V_1, V_2$  and  $\{x, y\}$  in  $G$  such that  $\{x, y\}$  is a cut edge separating  $V_1$  from  $V_2$  in  $G$  and both  $G[V_1]$  and  $G[V_2]$  are connected and contain a graph of  $B_{D-2}$  as a minor.*

**Proof:** If  $G$  contains a graph in  $4B_{D-2}$ , let  $V_1 \uplus V_2 \uplus V_3 \uplus V_4$  be a vertex partition of  $G$  such that, for all  $i$ , the graph  $G[V_i]$  is connected and contains a graph in  $B_{D-2}$  as a minor. Otherwise, let  $V_1 \uplus V_2 \uplus V_3 \uplus \{v\}$  be a vertex partitioning of  $G$  such that, for all  $i$ , the graph  $G[V_i]$  contains a graph in  $B_{D-2}$  as a minor.

Let  $G'$  denote the result of contracting each  $V_i$  into a single vertex. Since  $G'$  is outerplane, it cannot be a  $K_4$ . Thus, without loss of generality, the vertices corresponding to  $V_1$  and  $V_2$  are non-adjacent in  $G'$ . Thus, there is no edge between  $V_1$  and  $V_2$  in  $G$  and, since  $G$  is maximal, a cut-edge  $\{x, y\}$  exists in  $G[V_3 \uplus V_4]$  (or in  $G[V_3 \uplus \{v\}]$ ).  $\square$

By Observation 4, for each  $i \in \{1, 2\}$ , there is a triangulation  $G_i$  of  $G[V_i]$  that is a contraction of  $G$ . Analogously to Case 1, there is a parallel matching  $M'$  of  $G[V_1 \cup V_2]$  with  $|M'| \geq D - 1$ . Then,  $M := M' \cup \{\{x, y\}\}$  is a parallel matching in  $G$  with  $|M| \geq D$ . Then, the claim follows by Observation 3  $\square$

## B NP-hardness reduction

In this section we prove the NP-hardness of the following problem, which can be seen as a generalization of the PLANAR DIAMETER IMPROVEMENT problem.

### WEIGHTED 2-APEX DIAMETER IMPROVEMENT

**Input:** A 2-apex graph  $G = (V, E)$  and a weight function  $w : E \rightarrow \{1, 2, 5\}$ .  
**Output:** A set  $E' \subseteq (V \times V) \setminus E$  such that  $G' = (V, E \cup E')$  is a 2-apex graph.  
**Objective:** Minimize the diameter of  $G'$ , where  $w(e) = 1$  for all  $e \in E'$ .

Recall that for an integer  $p \geq 1$ , a graph  $G = (V, E)$  is a  $p$ -apex graph if there is a set  $A \subseteq V$  with  $|A| \leq p$  such that  $G \setminus A$  is planar. For an edge-weighted graph, the distance and the diameter are defined taking into account the weights of the edges.

**Theorem 4** WEIGHTED 2-APEX DIAMETER IMPROVEMENT is NP-hard.

**Proof:** We will prove that the decision version of WEIGHTED 2-APEX DIAMETER IMPROVEMENT is NP-complete. It is clear that the problem belongs to NP, as the (weighted) diameter of a graph can be computed in polynomial time. For the NP-hardness, we will reduce from the PLANAR 3-SAT problem where each variable occurs at most 3 times, which is known to be NP-complete [16]. Note that we can assume that each variable appears at least once in a positive literal and at least once in a negative literal, as otherwise we can set its value to true or false without affecting the satisfiability of the formula. More precisely, given a formula  $\phi$ , we will construct an edge-weighted 2-apex graph  $G$  that can be completed into a 2-apex graph  $G'$  of diameter at most 4 if and only if  $\phi$  is satisfiable. Let us now proceed to describe the reduction.

The formula  $\phi$  naturally defines a bipartite graph  $G_\phi$  with one vertex per variable and clause, and where a variable-vertex  $x_i$  is adjacent to a clause-vertex  $c_j$  if and only if the corresponding variable belongs to the corresponding clause. By hypothesis, the graph  $G_\phi$  is planar and each variable-vertex has degree at most 3. We fix an arbitrary plane embedding of  $G_\phi$ . Towards constructing  $G$ , we replace each variable-vertex  $x_i$  with the gadget depicted in Fig. 1, where the weights of the edges are also depicted in the figure, and where we assume for example that  $x_i$  appears negatively in  $c_{i_1}$  and positively in  $c_{i_2}$  and  $c_{i_3}$ .

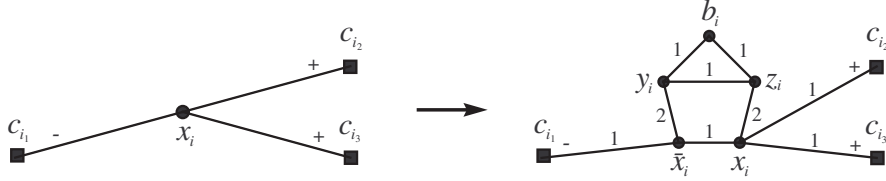


Figure 1: Gadget of the reduction in the proof of Theorem 4.

Since each variable  $x_i$  appears at most 3 times in  $\phi$ , in the embedding of  $G_\phi$  the edges between  $x_i$  and the clause-vertices corresponding to positive or negative variables (namely, *positive* and *negative* edges) appear consecutively in all circular orderings around  $x_i$ . Thus, in the gadget, all positive (resp. negative) edges can indeed be incident with  $x_i$  (resp.  $\bar{x}_i$ ) while preserving planarity<sup>3</sup>

We now triangulate arbitrarily the graph obtained so far with edges of weight 5, except for the 4-cycle defined by the vertices  $\{x_i, \bar{x}_i, y_i, z_i\}$  inside every gadget corresponding to a variable-vertex  $x_i$ . Finally, we add two adjacent new vertices  $a_1$  and

<sup>3</sup>Alternatively, the edge  $\{x_i, \bar{x}_i\}$  can be seen as the result of subdividing the unique edge incident to  $x_i$  in  $G$  whose sign (positive/negative) differs from the other two.

$a_2$  (the apices) and make them adjacent to all other vertices. The weights of these new edges are defined as follows. The edge between  $a_1$  and  $a_2$  has weight 5, and all other edges containing  $a_2$  have weight 2. The edges between  $a_1$  and the vertices  $b_i$  have weight 1, and all other edges containing  $a_1$  have weight 5. This completes the construction of  $G$ , which is clearly a 2-apex graph, as the removal of  $a_1$  and  $a_2$  yields a planar graph, relying on the planarity of  $G_\phi$ .

We claim that  $G$  can be completed into a 2-apex graph  $G'$  of diameter at most 4 if and only if  $\phi$  is satisfiable. Note that the only edges that can be added to  $G$  in order to build a 2-apex graph  $G'$  are those inside the 4-cycles  $\{x_i, \bar{x}_i, y_i, z_i\}$  for each variable-vertex  $x_i$ , namely either the edge  $\{x_i, y_i\}$  or the edge  $\{\bar{x}_i, z_i\}$  (see Fig. 1). Let us call these edges *potential* edges. The choice of the edge  $\{x_i, y_i\}$  (resp.  $\{\bar{x}_i, z_i\}$ ) corresponds to setting the variable  $x_i$  to true (resp. false) in the formula  $\phi$ . The reader can check that in  $G$  all pairs of vertices are within weighted distance at most 4 (via  $a_2$  or some  $b_i$ ) except for  $a_1$  and the clause-vertices  $c_j$ . It holds that each clause-vertex  $c_j$  is within distance at most 4 from  $a_1$  in  $G'$  if and only if for at least one of the literals  $\ell$  contained in the clause, the potential edge containing  $\ell$  has been chosen in order to construct  $G'$ , which translates into the fact that the clause is satisfied. Therefore,  $G$  can be completed into a 2-apex graph  $G'$  of diameter at most 4 if and only if all clauses of  $\phi$  can be simultaneously satisfied, as we wanted to prove.  $\square$